conditions of the formation and composition of the hydrate of phosphine.

I must express my best thanks to Professor Dewar, F.R.S., for his many kind and helpful suggestions, and also to the Master and Fellows of Christ's College, who have permitted me to retain my scholarship during the continuance of this work.

II. "On the Principal Electric Time-constant of a Circular Disk." By Horace Lamb, M.A., F.R.S., Professor of Pure Mathematics in the Owens College, Victoria University. Received March 29, 1887.

The time-constant for currents of any normal type in a given conductor is the time in which free currents of that type fall to $1/e$ of their original strength. In strictness there are for any conductor an infinite series of time-constants, corresponding to the various normal types, but in such a case as that of a coil of wire one of these is very great in comparison with the rest, which belong to types in which the current is in opposite directions in different parts of a section of the wire. And in all cases the time-constant corresponding to the most persistent type which can be present under given circumstances is, of course, the one which is most important from an experimental point of view.

A determination of the time-constants of a uniform circular disk would be of interest for two reasons: first, in relation to Arago's rotations, which are entirely due to the greater or less persistence of currents once started in the disk; and, secondly, in connexion with Professor Hughes's experiments with the induction balance, in which the disturbance produced in the field by the currents induced in metal disks (such as coins) was studied. Unfortunately, the mathematical problem thus suggested would seem to be difficult. Restricting ourselves, for simplicity, to cases where the currents flow in circles concentric with the disk, so that the problem is not complicated by the existence of an electric potential, then if ϕ be the current-function, the electric momentum at a distance r from the centre of the disk will be $-d\mathrm{P}/dr$, where P is the potential of an imaginary distribution of matter of density ϕ over the disk. Hence, if ρ' be the resistance per unit area, we have—

$$\rho'\frac{d\phi}{dr} = -\frac{d}{dt}\frac{d\mathrm{P}}{dr}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

In any normal type, ϕ and P will vary as $e^{-\lambda t}$, and, therefore—

$$\rho'\frac{d\phi}{dr} = \lambda\frac{d\mathrm{P}}{dr}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

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or, in the case of uniform resistance—

$$\rho'\phi = \lambda P + C$$

over the disk.

In the absence of a rigorous solution of this problem (which seems well worthy the attention of mathematicians), a good approximation to the principal time-constant may be obtained on the following principles:—[*]

1. An increase of resistance in any part of the disk will diminish the time-constant; and

2. If the time-constant be calculated on any arbitrary assumption as to the distribution of current, the result will be an *under*-estimate, and will, moreover, be a close approximation to the true value if the assumed law be not very wide of the mark, on account of the "stationary" property of the normal types.

Some distributions of density ϕ, and corresponding potentials P, convenient for our purpose, are obtained by considering the disk as a limiting form of heterogeneous ellipsoid, in which the surfaces of equal density are similar and coaxial ellipsoids.[†] If the density at any point Q in the interior of the ellipsoid—

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

be $$C(1-\theta^2)^n,$$

where θa, θb, θc are the semi-axes of the similar ellipsoid through Q, the corresponding potential at internal points will be—

$$C\,.\,\frac{\pi abc}{n+1}\int_0^\infty\left(1-\frac{x^2}{a^2+k}-\frac{y^2}{b^2+k}-\frac{z^2}{c^2+k}\right)^{n+1}\frac{dk}{\sqrt{\{(a^2+k)(b^2+k)(c^2+k)\}}}.$$

Putting $a = b$, and passing to the case of a disk, by putting $c = 0$, $2Cc = 1$, we find that to the surface-density—

$$\phi = \left(1-\frac{r^2}{a^2}\right)^{n+\frac{1}{2}}\int_0^{\frac{1}{2}\pi}\cos^{2n+1}\chi\,d\chi = \frac{\pi^{\frac{1}{2}}}{2}\cdot\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left(1-\frac{r^2}{a^2}\right)^{n+\frac{1}{2}}, \quad (3.)$$[‡]

corresponds, for points of the disk, the potential—

$$P = \frac{\pi a^2}{2(n+1)}\int_0^\infty\left(1-\frac{r^2}{a^2+k}\right)^{n+1}\frac{dk}{(a^2+k)k^{\frac{1}{2}}}$$

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[*] See Rayleigh's 'Sound,' §§ 88, 305, &c.

[†] See Ferrers, 'Quart. Journ. Math.,' vol. 14, p. 1.

[‡] In the electrical application we must suppose $n+\frac{1}{2}>0$.

$$= \frac{\pi a}{n+1}\int_0^{\pi}\left(1-\frac{r^2}{a^2}\sin^2\chi\right)^{n+1}d\chi$$

$$= \frac{\pi^2 a}{2(n+1)}\,\mathrm{F}\left(-n-1,\ \tfrac{1}{2},\ 1,\ \frac{r^2}{a^2}\right), \quad \dots\dots \quad (4.)$$

in the usual notation of hypergeometric series.

In the electrical problem, then, to a current-function of the form (3) corresponds the current-strength—

$$-\frac{d\phi}{dr} = \frac{\sqrt{\pi}}{a}\cdot\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\frac{r}{a}\left(1-\frac{r^2}{a^2}\right)^{n-\frac{1}{2}}, \quad \dots\dots\dots \quad (5.)$$

and the electric momentum—

$$-\frac{d\mathrm{P}}{dr} = \frac{\pi^2}{2}\cdot\frac{r}{a}\,\mathrm{F}\left(-n,\ \tfrac{3}{2},\ 2,\ \frac{r^2}{a^2}\right). \quad \dots\dots\dots \quad (6.)$$

The assumption (3) will correspond accurately to a normal type for a non-uniform disk, provided the law of resistance be properly adjusted. For (2) is satisfied if—

$$\frac{\rho'}{a}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\left(1-\frac{r^2}{a^2}\right)^{n-\frac{1}{2}} = 2\lambda\pi^{\frac{1}{2}}\int_0^{\pi}\left(1-\frac{r^2}{a^2}\sin^2\chi\right)^{n}\sin^2\chi\,d\chi$$

$$= \lambda\frac{\pi^{\frac{3}{2}}}{2}\,\mathrm{F}\left(-n,\ \tfrac{3}{2},\ 2,\ \frac{r^2}{a^2}\right),$$

that is, $$\rho' = \frac{4}{\pi}\rho_0'\int_0^{\frac{1}{2}\pi}\left(1-\frac{r^2}{a^2}\sin^2\chi\right)^{n}\sin^2\chi\,d\chi \div \left(1-\frac{r^2}{a^2}\right)^{n-\frac{1}{2}} \quad . \quad (7.)$$

$$= \rho_0'\mathrm{F}\left(-n,\ \tfrac{3}{2},\ 2,\ \frac{r^2}{a^2}\right)\div\left(1-\frac{r^2}{a^2}\right)^{n-\frac{1}{2}}, \quad \dots\dots \quad (8.)$$

where ρ_0' is the resistance at the centre; and the time-constant is—

$$\tau = \lambda^{-1} = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\,\frac{\pi^{\frac{3}{2}}a}{2\rho_0'}. \quad \dots\dots \quad (9.)$$

For $n = 0$, we have—

$$\rho' = \rho_0'(1-r^2/a^2)^{\frac{1}{2}},$$

$$\tau = \frac{\pi^2 a}{2\rho_0'}.$$

Since ρ' diminishes from the centre outwards, we see that $4{\cdot}935a/\rho'$ is a superior limit for a disk of uniform resistance ρ'.

For $n = \frac{1}{2}$,

$$\rho' = \frac{4}{\pi}\rho_0' \int_0^{\frac{1}{2}\pi} \left(1 - \frac{r^2}{a^2}\sin^2\chi\right)^{\frac{1}{2}} \sin^2\chi \, d\chi,$$

$$\tau = \frac{\pi a}{\rho_0'},$$

from which it appears that $3{\cdot}142a/\rho'$ is a superior limit for a uniform disk.

For $n = 1$,

$$\rho' = \rho_0'\left(1 - \tfrac{3}{4}\frac{r^2}{a^2}\right) \div \left(1 - \frac{r^2}{a^2}\right)^{\frac{1}{2}}, \quad \ldots \ldots \quad (10.)$$

$$\tau = \frac{\pi^2 a}{4\rho_0'}. \quad \ldots \ldots \ldots \ldots \quad (11.)$$

This case is remarkable as giving a resistance nearly uniform over the disk, except close to the edge, where it rapidly increases. The second column in the following table gives the resistance at various distances from the centre; the third column, the corresponding thicknesses in terms of the thickness at the centre, the material of the disk being supposed uniform.

r/a.	ρ'/ρ'_0.	Thickness.
0	1·000	1·000
0·1	0·998	1·002
0·2	0·990	1·010
0·3	0·978	1·023
0·4	0·960	1·041
0·5	0·938	1·066
0·6	0·913	1·096
0·7	0·886	1·129
0·8	0·867	1·154
0·9	0·900	1·111
0·95	1·035	0·966
0·99	1·878	0·532
1·00	∞	0

The minimum value of ρ' is $0{\cdot}8660\rho_0'$, corresponding to $r/a = 0{\cdot}8165$. Denoting this minimum by ρ_1', we find from (11)—

$$\tau = 2{\cdot}137\frac{a}{\rho_1'}.$$

This is an *inferior* limit to the value of τ for a disk of uniform resistance ρ_1'.

Some further information may be gathered from the second principle stated above. The electrokinetic energy of the system of currents defined by (3) is—

$$\mathrm{T} = \tfrac{1}{2}\int_0^a \frac{d\phi}{dr}\,\frac{d\mathrm{P}}{dr}\,.\,2\pi r\,dr$$

$$= \frac{\pi^{\frac{7}{2}}a}{4}\,\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\int_0^1 z(1-z)^{n-\frac{1}{2}}\,\mathrm{F}(-n,\,\tfrac{3}{2},\,2,\,z)\,dz.$$

The integral

$$= \Sigma_m\,.\,\frac{\Gamma(-n+m)\,.\,\Gamma(\frac{3}{2}+m)}{\Gamma(m+1)\Gamma(-n)\Gamma(\frac{3}{2})}\,.\,\frac{\Gamma(2)}{\Gamma(2+m)}\int_0^1 z^{m+1}(1-z)^{n-\frac{1}{2}}dz$$

$$= \Gamma(n+\tfrac{1}{2})\,.\,\Sigma_m\frac{\Gamma(-n+m)\Gamma(\frac{3}{2}+m)}{\Gamma(m+1)\Gamma(-n)\Gamma(\frac{3}{2})}\,\frac{1}{\Gamma(m+n+\frac{5}{2})}$$

$$= \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{5}{2})}\,\mathrm{F}_1(-n,\,\tfrac{3}{2},\,n+\tfrac{5}{2})$$

$$= \frac{\Gamma(n+\frac{1}{2})\Gamma(2n+1)}{\Gamma(2n+\frac{5}{2})\Gamma(n+1)}.$$

Hence
$$\mathrm{T} = \frac{\pi^{\frac{7}{2}}a}{4}\,.\,\frac{\Gamma(2n+1)}{\Gamma(2n+\frac{5}{2})}. \quad \dots\dots \quad (12.)$$

If the disk be of uniform resistance ρ', the dissipation is—

$$\mathrm{W} = \rho'\int_0^a\left(\frac{d\phi}{dr}\right)^2.\,2\pi r\,dr$$

$$= \pi^2\rho'\,.\,\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right\}^2\int_0^1 z(1-z)^{2n-1}dz$$

$$= \frac{\pi^2\rho'}{2n(2n+1)}\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right\}^2. \quad \dots\dots \quad (13.)$$

Introducing a time-factor $e^{-t/\tau}$, and supposing that the system of currents is constrained to remain of the type (3) during the decay, we find on equating the rate of diminution of the energy to the dissipation—

$$\tau = \frac{\pi^{\frac{3}{2}}a}{\rho'}\,\frac{\Gamma(2n+2)\{\Gamma(n+\frac{1}{2})\}^2}{\Gamma(n)\Gamma(n+1)\Gamma(2n+\frac{5}{2})}. \quad \dots\dots \quad (14.)$$

Any value of τ obtained from this formula will be an inferior limit to the true value. The following table gives the values of $\tau\rho'/a$ for different values of n:—

n.	$\tau\rho'/a$.
0·5	2·133
0·6	2·201
0·7	2·239
0·8	2·257
0·9	2·261
1·0	2·256
1·1	2·245
1·2	2·229
1·3	2·210
1·4	2·189
1·5	2·167
2·0	2·051

It appears that the value (14) of τ is a maximum for $n = 0·9$ about, and, hence, that the principal time-constant of a circular disk is not less than $2·26a/\rho'$. We have seen that the value of ϕ obtained by putting $n = 1$ in (3) must be a pretty fair representation of the most persistent type of free currents in a uniform disk, and the case of $n = 0·9$ will not be materially different. The "stationary" property already alluded to therefore warrants us in asserting that the value just given must be a close approximation to the truth. If δ be the thickness, ρ the specific resistance of the material, we may write our result thus—

$$\tau = 2·26\,\frac{a\delta}{\rho}.$$

For a disk of copper [$\rho = 1600$ C.G.S.], a decimetre in radius, and 2·5 mm. in thickness, this gives $\tau = 0·0035$ sec.

Addendum.—April 11, 1887.

In the above calculations it is assumed that the current-intensity is sensibly uniform throughout the thickness of the disk. This will be the case, at all events for a non-magnetisable substance, if the radius be a moderately large multiple of the thickness. To examine this point more closely, it will be sufficient to consider a simpler problem in which all the circumstances can be calculated with exactness. Let us suppose then that we have a system of free currents everywhere parallel to the axis of z in a stratum of conducting matter bounded by the planes $y = \pm\delta/2$. With the usual notation we shall have—

$$F = 0, \qquad G = 0,$$

$$a = \frac{dH}{dy}, \qquad b = -\frac{dH}{dx}, \qquad c = 0.$$

In the spaces on each side of the stratum—

$$\frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} = 0;$$

whilst in the conductor itself—

$$\frac{d^2H}{dx^2} + \frac{d^2H}{dy^2} = -4\pi\mu w.$$

The equation of electromotive force is—

$$\rho w = -\frac{dH}{dt} = \lambda H,$$

the time-factor, as before, being $e^{-\lambda t}$. Let us further assume that x enters into the value of H only through a factor sin mx. We shall then have, in the conductor—

$$\frac{d^2H}{dy^2} + k^2H = 0,$$

where $$k^2 = 4\pi\mu\lambda/\rho - m^2, \qquad \ldots\ldots\ldots \quad (15.)$$

and in the external spaces—

$$\frac{d^2H}{dy^2} - m^2H = 0.$$

The solutions of these equations, appropriate to our present problem, are

$$H = D \cos ky,$$

and $$H = D' e^{\mp my},$$

respectively, the upper or lower sign being taken in the latter expression according as y is positive or negative. At the surfaces of the conductor we must have—

$$a = \mu a', \qquad b = b',$$

when the accented letters relate to points just outside, the unaccented to points just inside. These conditions give—

$$kD \, . \sin\frac{k\delta}{2} = \mu m D' e^{-m\delta/2} \cos\frac{k\delta}{2},$$

$$D = D' e^{-m\delta/2},$$

whence
$$k\delta \, . \tan\frac{k\delta}{2} = \mu m\delta \, . \qquad (16.)$$

If $\mu = 1$, as we have supposed, and $m\delta$ is small, the principal root of this equation in $k\delta$ is small, and the current-intensity, which varies as $\cos ky$, will be nearly uniform throughout the thickness. The equation (16) then gives—

$$k^2\delta^2 = 2m\delta - \tfrac{1}{3}m^2\delta^2 + \&c.,$$

and therefore from (15)—

$$\tau = \lambda^{-1} = \frac{2\pi}{m\rho'}(1 - \tfrac{1}{3}m\delta + \&c.),$$

where $\rho' = \rho/\delta$. For the purpose of a rough comparison with our original problem we may suppose that π/m is comparable with R, the radius of the disk. It follows that the effect of replacing the actual disk, of finite thickness, by an infinitely thin disk of the same conductivity (per unit area) is to increase the time-constant by the fraction δ/R of itself, about.

In an *iron* plate, on the other hand, the current-intensity will fall off considerably from the median plane to the surface, unless the ratio δ/R be extremely small. For instance, if $\mu m\delta = \pi/2$, or say $\delta/R = 1/2\mu$, the principal root of (16) is $k\delta = \pi/2$, and the intensity at the surface is only 0·71 of its value in the median plane, although the thickness of the disk may perhaps not exceed one-thousandth of the radius. Again if, $m\delta$ being still small, $\mu m\delta$ is moderately large, we shall have $k\delta = \pi$, nearly, so that the current-intensity almost vanishes at the surface. In such a case—

$$\tau = 4\pi\mu/k^2\rho = 4\mu\delta^2/\pi\rho,$$

roughly. It will be seen that within certain limits (*e.g.*, if $\mu = 500$ and the lateral dimensions be not more than about 100 times the thickness) this result is independent of the size and shape of the plate. Under these circumstances, the value of τ for an iron plate ($\rho = 10{,}000$ C.G.S.) whose thickness is 2·5 mm. will be comparable with 0·003 sec.